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The dynamics of the weakly perturbed motion of a liquid-filled gyroscope and the control problem☆

A.A. Gurchenkov*, V.V. Korneyev, M.V. Nosov

Moscow, Russia

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ABSTRACT

The Cauchy problem for the motion of a dynamically symmetrical rigid body with a cavity, filled with an ideal liquid, which is perturbed from uniform rotation, is considered in a linear formulation. The problem of the simultanious solution of the equations of hydrodynamics and the mechanics of a rigid body is reduced to the solution of an eigenvalue problem which depends solely on the geometry of the cavity and the subsequent integration of a system of differential equations.

Sufficient conditions for stability, which are in agreement with the results of the solution of an analogous problem,[2] have been obtained earlier[1] in the problem of the stability of the rotational motions of a liquid-filled rigid body. These results supplement Zhukovskii's theorem.[3] The oscillations of a body with a liquid have been investigated where the quantities characterizing the body and the liquid were exponentially dependent on time.[4,5]

The quantity characterizing the perturbed motion of the body is assumed below to be an arbitrary function of time. The approach used allows one to obtain the dependence of the angular velocity in the perturbed motion on the external moment. The problem of the optimal control of the rotation of the body-liquid system with a mixed-type functional is raised.

1. The rotational motions of a body with a liquid-filled cavity

Consider the motion of a rigid body with a cavity Q, which is entirely filled with an ideal incompressible liquid of density ρ, in the field of mass forces with a potential U. We will write the equations of motion of the liquid and the boundary and initial conditions in the system of coordinates $Ox_1x_2x_3$ which is rigidly connected with the body:

$$\mathbf{w}_0 + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r} + 2\boldsymbol{\omega}\times\mathbf{V} + \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\rho^{-1}\nabla P - \nabla U, \ \operatorname{div}\mathbf{V} = 0 \qquad (1.1)$$

in the domain Q

$$\mathbf{n}\cdot\mathbf{V} = 0 \ \text{on} \ S, \quad \mathbf{V} = \mathbf{V}_0(\mathbf{r}) \ \text{when} \ t = 0$$

In the system of coordinates $Ox_1x_2x_3$, a time derivative is denoted by a dot, $\mathbf{w}_0$ is the absolute acceleration of the point O, $\boldsymbol{\omega}$ is the absolute angular velocity of the body, $\dot{\boldsymbol{\omega}}$ is its angular acceleration, r is the radius vector measured from the point O, $\mathbf{V}$ is the velocity of the liquid in the system of coordinates $Ox_1x_2x_3$, P is the pressure, S is the boundary of the domain Q and $\mathbf{n}$ is a unit vector of the outward normal to S.

We will write the angular momentum of the body with the liquid with respect to the centre of inertia O_1 of the whole system

$$\mathbf{K} = J\boldsymbol{\omega} + \rho\int_Q \mathbf{r}\times\mathbf{V}dQ; \quad J = J^1 + J^2 \qquad (1.2)$$

Here, J is the inertia tensor of the whole system with respect to the point O_1 which is put together from the inertia tensors of the body J^1 and the liquid J^2 with respect to the same point. A body filled with a liquid is a gyrostat, and the centre of inertia of the system O_1 is

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* Corresponding author.

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therefore fixed with respect to the system of coordinates $Ox_1x_2x_3$ and the tensors J^1, J^2, J are constant in this system of coordinates. The second term in equality (1.2), which is called the hydrostatic moment, is independent of the choice of the pole[5] and can be calculated with respect to the point O, as was done in equality (1.2).

We will write the equation of the moments with respect to the point O_1 in the system of coordinates $Ox_1x_2x_3$ associated with the body

$$\dot{\mathbf{K}} + \boldsymbol{\omega} \times \mathbf{K} = \mathbf{M}_1 \tag{1.3}$$

Here $\mathbf{M}_1$ is the principal moment, with respect to the point O_1, of all of the external forces acting on the body with the liquid.

Equations (1.1)–(1.3), together with the usual equations of motion of the centre of inertia, the kinematic relations and the initial conditions, completely describe the dynamics of the body with the liquid.

Suppose the unperturbed motion of the body with the liquid with respect to the centre of inertia O_1 is a rotation of the whole system about the O_1y_3 axis, which passes through the point O_1 parallel to the Ox_3 axis, with a constant angular velocity $\boldsymbol{\omega}_0$. In the unperturbed motion, we have

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 = \omega_0\mathbf{e}_3, \quad \mathbf{V} \equiv 0, \quad \mathbf{M}_1 = \boldsymbol{\omega}_0 \times J\boldsymbol{\omega}_0$$

where $\mathbf{e}_3$ is a unit vector along the Ox_3 axis.

We now consider the perturbed motion of the system. We put

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\Omega}(t), \quad P = \rho(-U - \mathbf{w}_0 \cdot \mathbf{r} + (\boldsymbol{\omega} \times \mathbf{r})^2/2 + p)$$
$$\mathbf{M}_1 = \boldsymbol{\omega}_0 \times J\boldsymbol{\omega}_0 + \mathbf{M} \tag{1.4}$$

and assume that the quantities $\boldsymbol{\Omega}$, $\mathbf{V}$, $\mathbf{M}$, p are of the first order of smallness in the perturbed motion.

Substituting relations (1.4) into Eqs. (1.1) and discarding the small higher-order terms, we reduce the problem of the motion of the liquid to the form

$$\frac{\partial\mathbf{V}}{\partial t} + 2\boldsymbol{\omega}_0 \times \mathbf{V} + \dot{\boldsymbol{\Omega}} \times \mathbf{r} = -\nabla p, \quad \operatorname{div}\mathbf{V} = 0 \text{ в } Q$$
$$\mathbf{n} \cdot \mathbf{V} = 0 \text{ on } S, \quad \mathbf{V} = \mathbf{V}_0(\mathbf{r}) \text{ when } t = 0 \tag{1.5}$$

Similarly, the equations of the body with the liquid take the form

$$J\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times J\boldsymbol{\omega}_0 + \boldsymbol{\omega}_0 \times J\boldsymbol{\Omega} + \rho\int_Q \mathbf{r} \times \dot{\mathbf{V}}dQ + \rho\int_Q \boldsymbol{\omega}_0 \times (\mathbf{r} \times \mathbf{V})dQ = \mathbf{M} \tag{1.6}$$

In problem (1.5), we now put $\boldsymbol{\Omega} = 0$ and consider the auxiliary problem of the oscillations of a liquid in a uniformly rotating vessel, the solution of which we shall seek in the form of harmonic oscillations

$$\mathbf{V} = \mathbf{u}(x_1, x_2, x_3)e^{i\lambda t}, \quad p = \varphi(x_1, x_2, x_3)e^{i\lambda t}$$

We therefore have the following problem

$$i\lambda_n\mathbf{u}_n + 2\boldsymbol{\omega}_0 \times \mathbf{u}_n + \nabla\varphi_n = 0, \quad \operatorname{div}\mathbf{u}_n = 0 \text{ в } Q; \quad \mathbf{u}_n \cdot \mathbf{n} = 0 \text{ on } S \tag{1.7}$$

For the function φ, we obtain the eigenvalue boundary-value problem

$$\Delta\varphi + \sigma^2\partial^2\varphi/\partial x_3^2 = 0 \text{ в } Q; \quad (L\nabla\varphi)\mathbf{n} = 0 \text{ on } S$$
$$L\mathbf{b} = \mathbf{b} + \sigma^2\mathbf{e}_3(\mathbf{e}_3 \cdot \mathbf{b}) + \sigma(\mathbf{b} \times \mathbf{e}_3) \tag{1.8}$$

The eigenfunctions $\varphi_n (n = 1,2,\ldots)$ of boundary value problem (1.8) correspond to the characteristic numbers $\sigma_n = -2i\omega_0/\lambda_n$ which densely fill the domain $\sigma_n \geq 1$ of the imaginary axis everywhere.[2] For practically interesting cavity shapes Q, the system of functions φ_n turns out to be a complete system of orthogonal functions.[6]

A dynamically symmetrical body with a cylindrical cavity

$$Q = \{(r, \theta, z); 0 \leq r \leq 1; 0 \leq \theta \leq 2\pi; -h \leq z \leq h\}$$

is next considered.

In this case, the solution of problem (1.8) in the cylindrical system of coordinates ($x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$) has the form

$$\varphi_n = \varphi_{lp}(r, \theta, z) = \frac{J_1(\xi_{lp}r)}{J_1(\xi_{lp})}\sin(k_lz)e^{i\theta}$$

$$k_l = \pi\frac{2l+1}{2h}, \quad \xi_{lp} = ik_l\sqrt{\sigma_{lp}^2 + 1}; \quad l = 0, 1, \ldots, \quad p = 1, 2, \ldots$$

The subscript n is all possible combinations of the ordinal numbers of the longitudinal and transverse harmonics l and p. The quantity ξ_{lp} is the p-th root of the equation $\xi J_1'(\xi)+i\sigma J_1(\xi)=0$. At the same time, the vector functions

$$\mathbf{u}_n = i\frac{L\nabla\varphi_n}{\lambda_n(1+\sigma_n^2)} \tag{1.9}$$

possess the property of orthogonality in the domain Q [6]

$$\int_Q \mathbf{u}_n\cdot\bar{\mathbf{u}}_m dQ = 0 \text{ when } n\neq m \tag{1.10}$$

The operation of complex conjugation is denoted by a bar.

We will seek a solution of Eqs. (1.5) using Bubnov's method. We will represent the velocity vector and the pressure in the form of an expansion in the vector eigenfunctions (1.9) of boundary-value problem (1.8)

$$\mathbf{V} = \sum_{n=1}^{\infty} S_n(t)\mathbf{u}_n(x_1,x_2,x_3), \quad p = \sum_{n=1}^{\infty} U_n(t)\varphi_n(x_1,x_2,x_3)$$

We will represent the term $\dot{\mathbf{\Omega}}\times\mathbf{r}$, associated with the motion of the body, in problem (1.5) in the form

$$\dot{\mathbf{\Omega}}\times\mathbf{r} = \sum_{n=1}^{\infty}\frac{\bar{\mathbf{a}}_n\cdot\dot{\mathbf{\Omega}}}{\mu_n^2}\mathbf{u}_n, \quad \mu_n^2 = \rho\int_Q|\mathbf{u}_n|^2dQ, \quad \mathbf{a}_n = \rho\int_Q\mathbf{r}\times\mathbf{u}_n dQ$$

and substitute these expansions into the first equation of (1.5), multiply scalarly by $\bar{\mathbf{u}}_m$ and then integrate over the volume of Q. The term $2\boldsymbol{\omega}_0\times\mathbf{u}_n$ in the resulting expression is found from the first equation of (1.7). By virtue of the orthogonality property (1.10), the first equation of (1.5) reduces to an infinite system of ordinary differential equations for the coefficients S_n

$$\mu_n^2(\dot{S}_n - i\lambda_n S_n) + \bar{\mathbf{a}}_n\cdot\dot{\mathbf{\Omega}} = 0, \quad n = 1, 2, \ldots \tag{1.11}$$

When account is taken of the expansions described above, the linearized equation of motion of the solid body will have the form

$$(J^0+J^1)\dot{\mathbf{\Omega}} + \mathbf{\Omega}\times(J^0+J^1)\boldsymbol{\omega}_0 + \boldsymbol{\omega}_0\times(J^0+J^1)\mathbf{\Omega} + \sum_{n=1}^{\infty}[\mathbf{a}_n\dot{S}_n + (\boldsymbol{\omega}_0\times\mathbf{a}_n)S_n] = \mathbf{M} \tag{1.12}$$

Here J^0 is the inertia tensor of the body without the liquid, J^1 is the inertia tensor of the solidified liquid and M is the moment of the external forces.

The system of equations of the perturbed motion of the body with a cavity filled with a liquid in the form (1.11) and (1.12) enables us to consider conditions when arbitrary control moments act on the body.

If the axis of rotation of the system in the unperturbed motion is simultaneously the axis of geometrical symmetry and of mass symmetry, the equations can be simplified considerably. In the case of a dynamically symmetrical body, the scalar equation of motion about the Ox_3 axis is separated from the remaining equations. The equations with respect to the transverse Ox_1 and Ox_2 axes are identical. In this case, Eqs. (1.11) and (1.12) can be written in the form

$$\begin{gathered} A\dot{\Omega} + i(C-A)\omega_0\Omega + \sum_{n=1}^{\infty}2a_n(\dot{S}_n - i\omega_0 S_n) = M \\ \mu_n^2(\dot{S}_n - i\lambda_n S_n) + a_n\dot{\Omega} = 0, \quad n = 1, 2, \ldots \end{gathered} \tag{1.13}$$

Here,

$$A = J_{11}^0 + J_{11}^1 = J_{22}^0 + J_{22}^1, \quad C = J_{33}^0 + J_{33}^1, \quad \Omega = \Omega_1 - i\Omega_2, \quad M = M_1 - iM_2$$

2. The stability of steady rotation of the body-liquid system

We will now consider the question of the stability of free rotation of the system. When $M=0$, the characteristic equation of system (1.13) has the form

$$A\eta + (C-A)\omega_0 - \eta(\eta-\omega_0)\sum_{n=1}^{\infty}\frac{E_n}{\eta-\lambda_n} = 0; \quad E_n = \frac{2a_n^2}{\mu_n^2} \tag{2.1}$$

For the stability of steady rotation, it is necessary that all of the roots of Eq. (2.1) should be real. For cavity shapes which are of practical interest, the convergence of the series in characteristic equation (2.1) follows from the form of the coefficients E_n, which enables us to consider approximations with a finite number of terms of the series. In the majority of cases, it is possible to confine overselves to the first approximation when, instead of the infinite sum in Eq. (2.1), just the leading term ($n = 1$) can be retained. In this case, the equations of the boundaries of the stability domain satisfy the equalities

$$C - A = -(\sqrt{E_1(1-\sigma_1)} \mp \sqrt{(A-E_1)\sigma_1})^2, \quad \sigma_1 = \lambda_1/\omega_0 \tag{2.2}$$

and the instability domain lies between the curves determined by equality (2.2). An analysis of the convergence of the series in Eq. (2.1), with the construction of the domains of stability of the free rotation of the body with the liquid, has been presented earlier.[6]

3. The equations of motion of the body-liquid system in an equivalent form

To solve the problem of the optimal control of the rotation of the body-liquid system, it is necessary to have an expression for the angular velocity Ω as a function of the control moment M.

We will denote the Laplace transform of the original $S(t)$ by $\hat{S}(p)$, that is,

$$\hat{S}(p) \equiv L[S(t)] = \int_0^\infty S(t)e^{-pt}dt$$

Performing a Laplace transformation on the equations of system (1.13), we express $\hat{S}_n$ from the n-th equation ($n > 1$) and, substituting it into the first equation, we obtain

$$\hat{\Omega}(p) = \hat{M}(p)\left[Ap + i(C-A)\omega_0 - p(p - i\omega_0)\sum_{n=1}^{\infty}\frac{E_n}{p - i\lambda_n}\right]^{-1} \tag{3.1}$$

For our subsequent analysis, we will retain a single term in the infinite sum and we have

$$\begin{aligned} \hat{\Omega}(p) &= \hat{M}(p)\hat{F}(p) \\ \hat{F}(p) &= (p - i\lambda_1)[(Ap + i(C-A)\omega_0)(p - i\lambda_1) - p(p - i\omega_0)E_1]^{-1} \end{aligned} \tag{3.2}$$

For the inverse Laplace transform, we make use of the convolution theorem and expansion theorem.[7] Then,

$$\Omega(t) = \int_0^t M(\tau)F(t-\tau)d\tau \tag{3.3}$$

Here

$$F(t) = \sum_{n=1}^{2}\frac{p_n - i\lambda_1}{2(A-E_1)p_n + i((C-A)\omega_0 + E\omega_0 - A\lambda_1)}e^{p_n t} \tag{3.4}$$

and p_1 and p_2 are the roots of the expression in the square brackets in the second equality of (3.2).

Expression (3.4) can be rewritten in the form

$$F(t) = (X_1 + iY_1)e^{p_1 t} + (X_2 + iY_2)e^{p_2 t}, \quad p_n = a_n + ib_n \tag{3.5}$$

where X_i and $Y_i (i = 1,2)$ are the real and imaginary parts of the fractional factor accompanying $e^{p_i t}$ in equality (3.4).

We recall that $\Omega = \Omega_1 - i\Omega_2$, $M = M_1 - iM_2$ and introduce the notation

$$\begin{aligned} P_k(t) &= \int_0^t [g_k(\tau)\cos(b_k(t-\tau)) + h_k(\tau)\sin(b_k(t-\tau))]e^{a_k(t-\tau)}d\tau \\ R_k(t) &= \int_0^t [h_k(\tau)\cos(b_k(t-\tau)) - g_k(\tau)\sin(b_k(t-\tau))]e^{a_k(t-\tau)}d\tau \\ g_k(\tau) &= M_1(\tau)X_k + M_2(\tau)Y_k, \quad h_k(\tau) = M_2(\tau)X_k - M_1(\tau)Y_k; \quad k = 1, 2 \end{aligned} \tag{3.6}$$

Taking account of formulae (3.4)–(3.6), we then have $\Omega_1 = P_1 + P_2$, $\Omega_2 = R_1 + R_2$. In the case of expression (3.3), we obtain a system of ordinary differential equations equivalent to it which, together with the initial conditions, having introduced the notation

$$\mathbf{x} = (\Omega_1, \Omega_2, P_1, P_2, R_1, R_2)$$

$$A = \left\| \begin{matrix} 0 & 0 & a_1 & a_2 & b_1 & b_2 \\ 0 & 0 & -b_1 & -b_2 & a_1 & a_2 \\ 0 & 0 & a_1 & 0 & b_1 & 0 \\ 0 & 0 & 0 & a_2 & 0 & b_2 \\ 0 & 0 & -b_1 & 0 & a_1 & 0 \\ 0 & 0 & 0 & -b_2 & 0 & a_2 \end{matrix} \right\|, \quad B = \left\| \begin{matrix} X_1+X_2 & Y_1+Y_2 \\ -Y_1-Y_2 & X_1+X_2 \\ X_1 & Y_1 \\ X_2 & Y_2 \\ -Y_1 & X_1 \\ -Y_2 & X_2 \end{matrix} \right\|$$

we write in the form

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{M}, \quad \mathbf{x}(0) = 0 \tag{3.7}$$

4. The problem of optimal control without constraints imposed on the control

We now consider the problem of the optimal control of system (3.7)

$$I(\mathbf{M}) = \sum_{k=1}^{2} (\Omega_k(T) - \Omega_k^0)^2 + \gamma \int_0^T (M_1^2(t) + M_2^2(t))dt \rightarrow \min$$

or, in vector form,

$$I(\mathbf{M}) = \|Z\mathbf{x}(T, \mathbf{M}) - \mathbf{b}\|_{E^6}^2 + \gamma \int_0^T \|\mathbf{M}\|_{E^2}^2 dt \rightarrow \min \tag{4.1}$$

where $\mathbf{M} = (M_1, M_2)^T$ is the required control, γ is a real positive number, Z is a 6×6 matrix with the non-zero elements $z_{1,1} = z_{2,2} = 1$, $\boldsymbol{b} = (\Omega_1^0, \Omega_2^0, 0, 0, 0, 0)^T$, Ω_1^0, Ω_2^0 are specified real numbers and $||\boldsymbol{x}||_{E^n} = \sqrt{\sum_{k=1}^{n} x_k^2}$ is the Euclidean norm of the n-th component vector x.

In the case considered, the functional (4.1) is convex as the sum of convex and strongly convex functionals for the linear system (3.7). The Hamilton–Pontryagin function for problem (3.7), (4.1) has the form

$$H = -\gamma\mathbf{M}^2 + \boldsymbol{\psi} \cdot (A\mathbf{x} + B\mathbf{M}) \tag{4.2}$$

The adjoint problem is then obtained in the form

$$\dot{\boldsymbol{\psi}}(t) = -A^T\boldsymbol{\psi}(t); \quad \psi_k(T) = \begin{cases} -2(\Omega_k(T) - \Omega_k^0), & k = 1, 2 \\ 0, \quad k = 3, 4, 5, 6 \end{cases} \tag{4.3}$$

Its solution is written as follows:

$$\begin{aligned} &\psi_k = -2(\Omega_k(T) - \Omega_k^0), \quad k = 1, 2; \quad \psi_3 = \zeta_{112}, \quad \psi_4 = \zeta_{122}, \quad \psi_5 = \zeta_{211}, \quad \psi_6 = \zeta_{221} \\ &\zeta_{pqr} = 2(\Omega_p(T) - \Omega_p^0) - 2e^{-a_q(t-T)}[(\Omega_p(T) - \Omega_p^0)\cos(b_q(t-T)) + (-1)^r(\Omega_r(T) - \Omega_r^0) \times \\ &\times \sin(b_q(t-T))], \\ &p, q, r = 1, 2 \end{aligned} \tag{4.4}$$

We will obtain expressions for the optimal control when there are no constraints using the solution (4.4) of the adjoint system (4.3) and the condition $\partial H/\partial\mathbf{M} = 0$. We have

$$M_k^*(t) = -\frac{1}{\gamma}[(\Omega_k(T) - \Omega_k^0)\xi(t-T) + (-1)^{k+1}(\Omega_{3-k}(T) - \Omega_{3-k}^0)\eta(t-T)], \; k = 1, 2 \tag{4.5}$$

where

$$\begin{aligned} \xi(z) &= \sum_{k=1}^{2} (X_k\cos b_k z + Y_k\sin b_k z)e^{-a_k z} \\ \eta(z) &= \sum_{k=1}^{2} (X_k\sin b_k z - Y_k\cos b_k z)e^{-a_k z} \end{aligned} \tag{4.6}$$

Use is made of formulae (3.3)–(3.6) for determining $\Omega_1(T)$, $\Omega_2(T)$.

Hence, solving the linear system of integral equations (4.5), we finally obtain the solution of the problem in the analytical form

$$M_k^*(t) = \frac{1}{\gamma+\theta}\{\Omega_k^0\xi(t-T)+(-1)^{k+1}\Omega_{3-k}^0\eta(t-T)\}, \quad k = 1, 2$$

$$\theta = \int_0^T(\xi^2(\tau-T)+\eta^2(\tau-T))d\tau$$

$$\Omega_k^*(t) = \int_0^t\{M_k^*(\tau)\xi(t-\tau)-(-1)^{k+1}M_{3-k}^*(\tau)\eta(t-\tau)\}d\tau, \quad k = 1, 2 \tag{4.7}$$

5. The problem of optimal control with inequality-type

We now consider the problem of the optimal control of system (3.7)

$$I(\mathbf{M}) = \|Z\mathbf{x}(T,\mathbf{M})-\mathbf{b}\|_{E^6}^2+\gamma\int_0^T\|\mathbf{M}\|_{E^2}^2dt \to \min; \quad |M_1(t)| \le C_1, \quad |M_2(t)| \le C_2 \tag{5.1}$$

where Ω_1^0, Ω_2^0, C_1, C_2 are specified real numbers and the remaining notation is the same as in Section 4.

The set of permissible controls (5.1) is convex in the space of the vector functions, which are square integrable and functional (4.1) is convex on the set (5.1).

The numerical Chernous'ko–Krylov method,[8,9] which enables one to effectively separate out discontinuities, is used to solve the problem.

The behaviour of the optimal control moment **M** and the angular velocity **Ω** in the case of the given moment is shown in Fig. 1 for constraints of the type (5.1) for a weightless cylindrical shell of height H = 0.5, 0.7 and radius of the base equal to unity, is entirely filled with an ideal liquid. The time T = 1 is the characteristic time for the rotation of the shell about its axis, Ω_1^0, $\Omega_2^0 = 0.01$, and the coefficients $\gamma = 1$ $C_1 = C_2 = 0.01$.

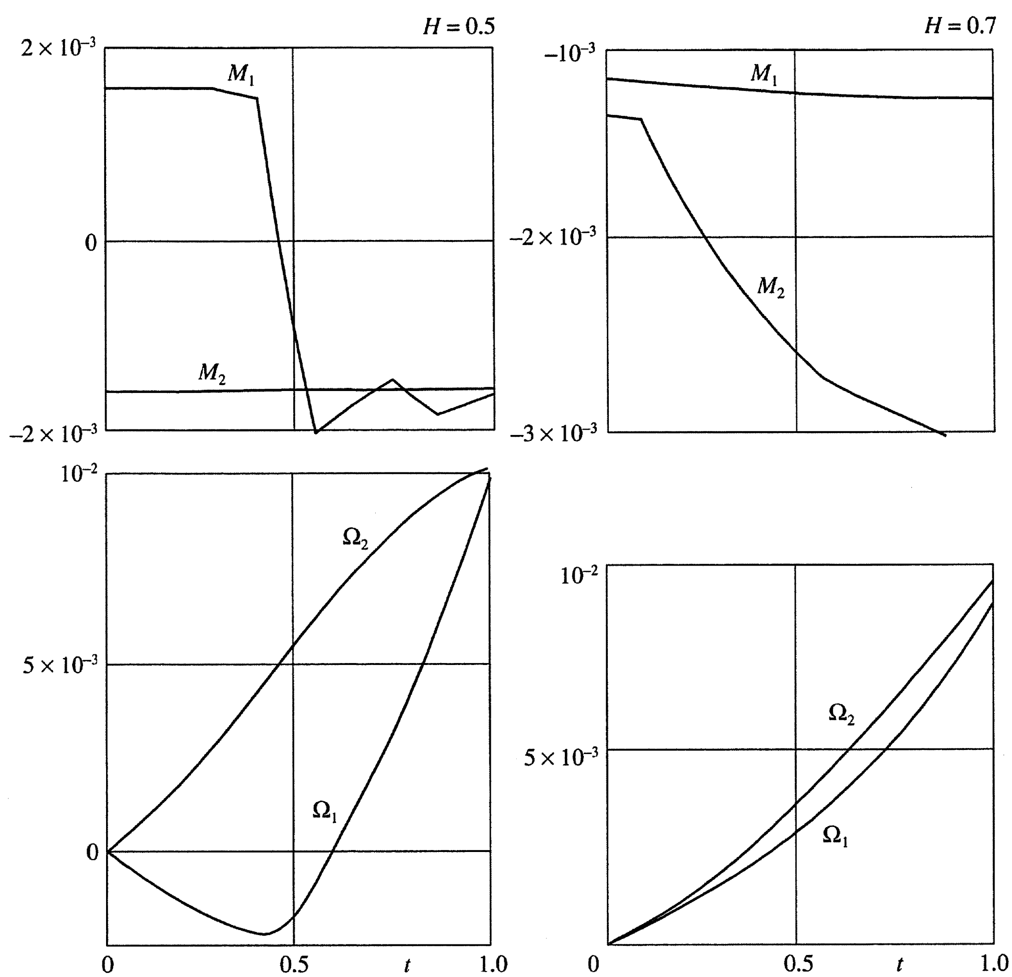

Fig. 1.

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